MATH 121A Prep: Diagonalization

Facts to Know:

Diagonalizable: A matrix A is called diagonalizable if there are two matrices D and P such that D is diagonal and P is invertible such that $A = PDP^{-1} \longleftrightarrow P^{-1}AP = D$

* An $n \times n$ matrix with n distinct eigenvalues is always diagonalizable.

Diagonalization Process:

1. Find all eigenvalues of A
2. Find an eigenvector corresponding to each eigenvalue
3. D has the eigenvalues down the diagonal
4. P has columns equal to the eigenvector corresponding to the eigenvalue in the same column of D
5. Invert P

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

$$[P \,\vdots\, Id] \xrightarrow{\text{row reduce}} [Id \,\vdots\, P^{-1}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Examples:

1. If $A = PDP^{-1}$, what is A^n?

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \text{ then } D^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

$$A^2 = (PD\underbrace{P^{-1})(P}_{Id}DP^{-1}) = PDDP^{-1} = P\underline{D^2}P^{-1}$$

$$A^3 = (PD\underbrace{P^{-1})(P}D^2P^{-1}) = PDD^2P^{-1} = PD^3P^{-1}$$

$$\vdots$$

$$\boxed{A^n = P\underline{D^n}P^{-1}}$$

2. Diagonalize the matrix $\begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}$.

1. Eigenvalues $0 = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b-a \\ 0 & b-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda)$

$\lambda = a, \ \lambda = b \qquad D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$\lambda = a: (A-\lambda I)\vec{v} = \vec{0} \quad \left[\begin{array}{cc|c} 0 & b-a & 0 \\ 0 & b-a & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_1 free, $x_2 = 0 \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ eigenvector

$\lambda = b: \left[\begin{array}{cc|c} a-b & b-a & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ x_2 free, $x_1 = x_2$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix} = PDP^{-1}$

3. Use the diagonalized form to calculate A^n.

$$A^n = PD^nP^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^n & -a^n \\ 0 & b^n \end{bmatrix}$$

$$= \begin{bmatrix} a^n & b^n - a^n \\ 0 & b^n \end{bmatrix} = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}^n$$